

# DEGENERATING THE JACOBIAN: THE NÉRON MODEL VERSUS STABLE SHEAVES

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**ABSTRACT.** We provide sufficient conditions for the line bundle locus in a family of compact moduli spaces of pure sheaves to be isomorphic to the Néron model. The result applies to moduli spaces constructed by Eduardo Esteves and Carlos Simpson, extending results of Busonero, Caporaso, Melo, Oda, Seshadri, and Viviani.

## 1. INTRODUCTION

**1.1. Background.** This paper relates two different approaches to extending families of Jacobian varieties. Recall that if  $X_0$  is a smooth projective curve of genus  $g$ , then the associated Jacobian variety is a  $g$ -dimensional smooth projective variety  $J_0$  that can be described in two different ways: as the universal abelian variety that contains  $X_0$  (the Albanese variety), and as the moduli space of degree 0 line bundles on  $X_0$  (the Picard variety). If  $X_U \rightarrow U$  is a family of smooth, projective curves, then the Jacobians of the fibers fit together to form a family  $J_U \rightarrow U$ . In this paper,  $U$  will be an open subset of a smooth curve  $B$  (or, more generally, a Dedekind scheme), and we will be interested in extending  $J_U$  to a family over  $B$ . Corresponding to the two different ways of describing the Jacobian (Albanese vs. Picard) are two different approaches to extending the family  $J_U \rightarrow U$ .

Viewing the Jacobian as the Albanese variety, it is natural to try to extend  $J_U \rightarrow U$  by extending it to a family of group varieties over  $B$ . Néron showed that this can be done in a canonical way in [Nér64]. He worked with an arbitrary family of abelian varieties  $A_U \rightarrow U$  and proved that there is a unique  $B$ -smooth group scheme  $N := N(A_U) \rightarrow B$  extending  $A_U \rightarrow U$  which is universal with respect to a natural mapping property. This scheme is called the Néron model. Arithmetic geometry has seen the use of the Néron model in a number of important results (e.g., [Maz72], [Maz77], [MW84], [Gro90]). The Néron model of a Jacobian variety plays a particularly prominent role, and an alternative description of this scheme in

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terms of the relative Picard functor was given by Raynaud ([Ray70]). We primarily work with Raynaud's description, which is recalled in Section 2.

An alternative approach, suggested by viewing the Jacobian as the Picard variety, is to extend  $J_U \rightarrow U$  as a family of moduli spaces of sheaves. This approach was first proposed by Mayer and Mumford in [MM64]. Typically, one first extends  $X_U \rightarrow U$  to a family of curves  $X \rightarrow B$  and then extends  $J_U$  to a family  $\bar{J} \rightarrow B$  with the property that the fiber over a point  $b \in B$  is a moduli space of sheaves on  $X_b$  parameterizing certain line bundles, together with their degenerations. In this paper, we show that the line bundle locus  $J$  in  $\bar{J}$  is canonically isomorphic to the Néron model for some schemes  $\bar{J}$  constructed in the literature.

To state this more precisely, we need to specify which schemes  $\bar{J}$  we consider. The problem of constructing such a family of moduli spaces has been studied by many mathematicians who have constructed different compactifications (e.g., [Ish78], [D'S79], [OS79], [AK80], [Cap94], [Sim94], [Pan96], [Jar00], [Est01]). Many of the difficulties to performing such a construction arise from the fact that, when  $X_b$  is reducible, the degree 0 line bundles on a fiber  $X_b$  do not form a bounded family.

For simplicity, assume the residue field  $k(b)$  is algebraically closed and  $X_b$  is reduced with components labeled  $X_1, \dots, X_n$ . Given a line bundle  $\mathcal{M}$  of degree 0 on  $X_b$ , the sequence  $(\deg(\mathcal{M}|_{X_1}), \dots, \deg(\mathcal{M}|_{X_n}))$  is called the multi-degree of  $\mathcal{M}$ . This sequence must sum to 0, but may otherwise be arbitrary, which implies unboundedness. A bounded family can be obtained by fixing the multi-degree, and typically the scheme  $\bar{J}$  is defined so that it parameterizes (possibly coarsely) line bundles (and their degenerations) that satisfy a numerical condition on the multi-degree. This paper focuses on constructions given by Simpson ([Sim94]) and by Esteves ([Est01]), which we now describe in more detail.

For the Simpson moduli space, the numerical condition imposed on line bundles is slope semi-stability with respect to an auxiliary ample line bundle. This condition arises from the method of construction: the moduli space is constructed using Geometric Invariant Theory, and slope stability corresponds to GIT stability. In general, the Simpson moduli space is a coarse moduli space in the sense that non-isomorphic sheaves may correspond to the same point of the space, but it contains an open subscheme (the stable locus) that is a fine moduli space, and we will work exclusively with this locus. Families of moduli spaces of semi-stable sheaves have been constructed for arbitrary families of projective schemes, but we will only be concerned with the case of families of curves.

The moduli spaces of Esteves parameterize sheaves that are  $\sigma$ -quasi-stable. Like slope stability,  $\sigma$ -quasi-stability is a numerical condition on the multi-degree, but it is defined in terms of an auxiliary vector bundle  $\mathcal{E}$  and a section  $\sigma$ , rather than an ample line bundle. The moduli spaces are constructed for families over an arbitrary locally noetherian base, but

strong conditions are required of the fibers: they must be geometrically reduced. The space is constructed as a closed subspace of a (non-noetherian, non-separated) algebraic space that was constructed in [AK80]. For nodal curves, the authors of [MV10] describe the relation between the Esteves moduli spaces and the Simpson moduli spaces. However, here we treat these moduli spaces separately.

For a discussion of the relation between these moduli spaces and other moduli spaces constructed in the literature, the reader is directed to [Ale04] and [CMKV11, Sect. 2]. The reader familiar with the work of Caporaso is warned that there is one potential source of confusion. In [Cap94], the compactified Jacobian associated to a stable curve  $X$  parameterizes pairs  $(Y, L)$  consisting of a line bundle  $L$  on a nodal curve  $Y$  stably equivalent to  $X$  that satisfies certain conditions. The line bundle locus  $J$  that we study corresponds to the locus parameterizing pairs  $(Y, L)$  with  $X = Y$ .

**1.2. Main Result.** The main result of this paper relates the line bundle locus in a proper family of moduli spaces of sheaves to the Néron model of the Jacobian:

**Theorem A.** *Let  $f: X \rightarrow B$  be a family of geometrically reduced curves over a Dedekind scheme with  $X$  regular. Define the following schemes:*

- (1)  $\mathcal{J} \subset \bar{\mathcal{J}}$  *is the line bundle locus in one of the following families of moduli spaces:*
  - *the Esteves compactified Jacobian or*
  - *the Simpson compactified Jacobian associated to a polarization such that slope semi-stability coincides with slope stability;*
- (2)  $N$  *is the Néron model of the Jacobian of the generic fiber  $X_\eta$ .*

*Then*

$$\mathcal{J} = N.$$

Theorem A is the combination of Corollaries 4.2 and 4.5, which themselves are consequences of Theorem 3.10. Theorem 3.10 is quite general, and we expect that it applies to many other fine moduli spaces of sheaves (but NOT coarse ones). In particular, Theorem 3.10 applies to families of curves with possibly non-reduced fibers, though then general results asserting the existence of a suitable moduli space are unknown (but see Sect. 4.3 for some simple examples).

The arithmetically-inclined reader should note that Theorem A and the results later in this paper do not place any hypotheses on the base Dedekind scheme  $B$ . In particular, we do not assume that the residue fields are perfect. The author was initially surprised by this as there is a body of work (e.g. [LLR04], [Ray70]) showing that various pathologies can arise when  $k(b)$  fails to be perfect.

Theorem A has interesting consequences for both the Néron model and the compactified Jacobian. One consequence of the theorem is that Néron models of Jacobians can often be constructed over high dimensional bases.

The Néron model of an abelian variety is only defined over a (regular) 1-dimensional base  $B$ , but no such dimensional hypotheses are need to apply the existence results from [Sim94] and [Est01]. At the end of Section 4.3, we examine a family  $J \rightarrow \mathbb{P}^2$  over the plane with the property that  $\mathbb{P}^2$  is covered by lines  $C$  such that the restriction  $J_C$  is the Néron model of its generic fiber. Surprisingly, while the Néron models fit into a 2-dimensional family, their group structure does not.

Theorem A also has interesting consequences for the moduli spaces of Esteves and Simpson. Indeed, if  $f: X \rightarrow B$  is a family of curves satisfying the hypotheses of the theorem, then both the Esteves Jacobians  $J_{\mathcal{E}}^g$  and the Simpson Jacobians  $J_{\mathcal{L}}^0$  (for  $\mathcal{L}$  as in the hypothesis) are independent of the particular polarizations, and every such Simpson Jacobian is isomorphic to every Esteves Jacobian. This is not immediate from the definitions. At the end of Section 4.1, we discuss this fact in greater detail and pose a related question.

**1.3. Past Results.** Certain cases of Theorem A were already known. In his thesis ([Bus08]), Simone Busonero established Theorem A for certain Esteves Jacobians. The thesis is unpublished, but a different proof using similar techniques that extends the result to the Simpson moduli spaces can be found in the pre-print [MV10, Thm. 3.1] by Melo and Viviani. They prove Theorem A when the fibers of  $f$  are nodal and  $X$  is regular. We do not discuss the Caporaso universal compactified Jacobian here, but the relation between that scheme and the Néron model was described by Caporaso in [Cap08a], [Cap08b], and [Cap10] (esp. [Cap10, Thm. 2.9]). Earlier still, Oda and Seshadri related their  $\phi$ -semi-stable compactified Jacobians, also not discussed here, to Néron models ([OS79, Cor. 14.4]). In each of those papers, an important step in the proof is a combinatorial argument establishing that, for example, the natural map from the set of  $\sigma$ -quasi-stable multi-degrees to the degree class group is a bijection.

The proof given here does not use any combinatorics, and the idea can be described succinctly. Consider the special case where  $B := \text{Spec}(\mathbb{C}[[t]])$ , which is a strict henselian discrete valuation ring with algebraically closed residue field. There is a natural map  $J \rightarrow N$  to the Néron model coming from the universal property of  $N$ , and an application of Zariski's Main Theorem shows that this morphism is an open immersion. Thus, the only issue is set-theoretic surjectivity. Because  $B$  is henselian, every point on the special fiber of  $N$  is the specialization of a section, so surjectivity is equivalent to the surjectivity of the map  $J(\mathbb{C}[[t]]) \rightarrow J(\text{Frac } \mathbb{C}[[t]])$  that sends a section to its restriction to the generic fiber. A given point  $p \in J(\text{Frac } \mathbb{C}[[t]])$  may be extended to a section  $\sigma \in \bar{J}(\mathbb{C}[[t]])$  of  $\bar{J}$  by the valuative criteria. As  $\bar{J}$  is a fine moduli space,  $\sigma$  corresponds to a family of rank 1, torsion-free sheaves, which in fact must be a family of line bundles because  $X$  is factorial. We may conclude that  $\sigma \in J(\mathbb{C}[[t]])$ , yielding the result.

**1.4. Questions.** It would be interesting to know when a Simpson Jacobian  $J_{\mathcal{L}}^0$  satisfying the hypotheses of Theorem A exists; that is, given a family  $f: X \rightarrow B$ , does there exist an ample line bundle  $\mathcal{L}$  such that every  $\mathcal{L}$ -slope semi-stable sheaf of degree 0 is stable? This question seems most interesting when the geometric fibers of  $f$  are non-reduced. We briefly survey the literature on this topic at the end of Section 4.2.

More generally, given a family  $f: X \rightarrow B$ , it would be desirable to have a description of the maximal subfunctors of the degree 0 relative Picard functor  $P^0$  representable by a separated  $B$ -scheme. We discuss this question in Section 4.3, where we analyze the simple case of genus 1 curves.

**1.5. Organization.** We end this introduction with a few technical remarks about the paper. The moduli spaces of sheaves that we consider are moduli spaces of pure sheaves. On a curve, a coherent sheaf is pure if and only if it is Cohen–Macaulay. This condition is also equivalent to the condition of being torsion-free in the sense of elementary algebra when the curve is integral, and the term “torsion-free” is sometimes used in place of “pure.”

The term “family of curves” only refers to families with geometrically irreducible generic fibers. This is done to avoid notational complications concerning multi-degrees. Families of curves are required to be proper but not projective. A family of curves over a Dedekind scheme can fail to be projective (e.g. [AK80, 8.10]), but projectivity is automatic if the local rings of the total space are factorial, which is the main case of interest. (See Prop. 4.1.)

We prove the main results for families over a base scheme  $S$  that is the spectrum of a strict henselian discrete valuation ring rather than a more general Dedekind scheme. Doing so lets us make section-wise arguments because a smooth family of a henselian base admits many sections. Furthermore, this is not a real restriction: results over a general Dedekind base can be deduced by passing to the strict henselization.

The body of the paper is organized as follows. In Section 2, we review Raynaud’s construction of the maximal separated quotient. We then relate this scheme to a general moduli space of line bundles satisfying some axioms in Section 3. Finally, we describe some schemes that satisfy these axioms in the final section, Section 4.

## CONVENTIONS

**1.1.** *The symbol  $X_T$  denotes the fiber product  $X \times_S T$ .*

**1.2.** *The letter  $S$  denotes the spectrum of a strict henselian discrete valuation ring with special point 0 and generic point  $\eta$ .*

**1.3.** *A curve over a field  $k$  is a proper  $k$ -scheme  $f_0: X_0 \rightarrow \text{Spec}(k)$  that is geometrically connected and of pure dimension 1.*

**1.4.** *If  $B$  is a scheme, then a family of curves over  $B$  is a proper, flat morphism  $f: X \rightarrow B$  whose fibers are curves and whose geometric generic fibers are irreducible.*

**1.5.** *If  $f: Y \rightarrow B$  is a finitely presented morphism, then we write  $Y^{sm} \subset Y$  for the smooth locus of  $f$ .*

**1.6.** *A coherent module  $I_0$  on a noetherian scheme  $X_0$  is rank 1 if the localization of  $I_0$  at  $x$  is isomorphic to  $\mathcal{O}_{X_0,x}$  for every generic point  $x$ .*

**1.7.** *A coherent module  $I_0$  on a noetherian scheme  $X_0$  is pure if the dimension of  $\text{Supp}(I_0)$  equals the dimension of  $\text{Supp}(J_0)$  for every non-zero subsheaf  $J_0$  of  $I_0$ .*

**1.8.** *If  $X_0 \rightarrow \text{Spec}(k)$  is proper, then the degree of a coherent  $\mathcal{O}_{X_0}$ -module  $\mathcal{F}$  is defined by  $\deg(\mathcal{F}) := \chi(\mathcal{F}) - \chi(\mathcal{O}_X)$ .*

## 2. RAYNAUD'S MAXIMAL SEPARATED QUOTIENT

We begin by reviewing Raynaud's construction of the Néron model of a Jacobian and, more generally, the maximal separated quotient of the relative Picard functor ([Ray70]). Much of this material is also treated in [BLR90, Chap. 9].

Let  $S$  be a strict henselian discrete valuation ring with generic point  $\eta$  and special point  $0$ . Given a family of curves  $f: X \rightarrow S$ , the **relative Picard functor**  $P$  of  $f$  is defined to be the étale sheaf  $P: S\text{-Sch} \rightarrow \text{Grp}$  associated to functor

$$(2.1) \quad T \mapsto \text{Pic}(X_T).$$

Here  $\text{Pic}(X_T)$  is the set of isomorphism classes of line bundles on  $X_T$ . Raynaud actually defines  $P$  to be the associated fppf sheaf but then observes that this is the same as the associated étale sheaf ([Ray70, 1.2]; see also [Kle05, Rmk. 9.2.11]). The fibers of  $P$  are representable by group schemes locally of finite type, and  $P$  itself is representable by an algebraic space if and only if  $f$  is cohomologically flat ([Ray70, Thm. 5.2]). Regardless of the representability properties,  $P$  is locally finitely presented and formally  $S$ -smooth.

Inside of  $P$ , we may consider the subfunctor  $E: S\text{-Sch} \rightarrow \text{Grp}$  that is defined to be the scheme-theoretic closure of the identity section. This is the fppf subsheaf of  $P$  generated by the elements  $g \in P(T)$ , where  $T \rightarrow S$  is flat and  $g_\eta \in P(T_\eta)$  is the identity element. When  $P$  is a scheme, this coincides with the usual notion of closure. The representability properties of  $E$  are similar to those of  $P$ : the fibers of  $E$  are group schemes locally of finite type and  $E$  is representable by an algebraic space precisely when  $f$  is cohomologically flat ([Ray70, Prop. 5.2]). When representable,  $E \rightarrow S$  is an étale  $S$ -group space; in general, the generic fiber of  $E$  is the trivial group scheme, and the special fiber is a group scheme of dimension equal to  $h^0(\mathcal{O}_{X_0}) - h^0(\mathcal{O}_{X_\eta})$ .

When  $E$  is not the trivial  $S$ -group scheme,  $P$  does not satisfy the valuative criteria of separatedness. We can, however, form the **maximal separated quotient**  $Q: S\text{-Sch} \rightarrow \text{Grp}$  of  $P$ . By definition, this is the fppf quotient sheaf  $Q := P/E$ . The maximal separated quotient  $Q$  is always representable by a scheme that is  $S$ -smooth, separated, and locally of finite type ([Ray70, Thm. 4.1.1, Prop. 8.0.1]). Rather than working directly with  $Q$ , we shall primarily work with a slightly smaller subfunctor  $Q^\tau: S\text{-Sch} \rightarrow \text{Grp}$ , which we now define.

Suppose generally that  $B$  is a scheme and  $G: B\text{-Sch} \rightarrow \text{Grp}$  is a  $B$ -group functor whose fibers are representable by group schemes locally of finite type. For every point  $b \in B$ , we may form the identity component  $G_b^\circ \subset G_b$  and the component group  $G_b/G_b^\circ$ . The subgroup functor  $G^\tau \subset G$  is defined to be the subfunctor whose  $T$ -valued points are elements  $g \in G(T)$  with the property that, for every  $t \in T$  mapping to  $b \in B$ , the element  $g_t \in G_b(k(t))$  maps to a torsion element of  $G_b/G_b^\circ(k(t))$ . If we instead require that  $g_t$  maps to the identity element, then we obtain the subgroup functor  $G^\circ \subset G$ . Let us examine these constructions when  $B$  equals  $S$  and  $G$  equals  $P$  or  $Q$ .

The functors  $P^\circ$  and  $P^\tau$  coincide, and this common functor is the étale sheaf associated to the assignment sending  $T$  to the set of isomorphism classes of line bundles on  $X_T$  that fiber-wise have multi-degree 0. From this description, it is easy to see that  $P^\circ = P^\tau \subset P$  is an open subfunctor. Another open subfunctor of  $P$  is the subfunctor parameterizing line bundles on  $X_T$  with fiber-wise degree 0, which we denote by  $P^0$ . It is typographically difficult to distinguish between  $P^0$  and  $P^\circ$ , but we will not make use of  $P^\circ$  in this paper, so this should not cause confusion.

The functors  $Q^\circ$  and  $Q^\tau$  are different in general. They are, however, both open subfunctors of  $Q$  ([FGA, Thm. 1.1(i.i), Cor. 1.7]). In particular, they are both representable by smooth and separated  $S$ -group schemes that are locally of finite type. In fact, both schemes are of finite type over  $S$  as their fibers are easily seen to have a finite number of connected components. The condition that  $Q^\tau \subset Q$  is a closed subscheme is important, but slightly subtle. A characterization of this condition is given by Prop. 8.1.2(iii) of [Ray70]; one sufficient (but not necessary) condition for  $Q^\tau \subset Q$  to be closed is that the local rings of  $X$  are factorial.

The factoriality condition is also almost sufficient to ensure that  $Q^\tau$  is the Néron model of its generic fiber. Suppose that the generic fiber of  $f$  is smooth, so the generic fiber of  $Q^\tau \rightarrow S$  is an abelian variety, and thus it makes sense to speak of the Néron model  $N := N(Q_\eta^\tau)$ . By the universal property, there is a unique morphism  $Q^\tau \rightarrow N$  that is the identity on the generic fiber. Theorem 8.1.4 of [Ray70] states that if the local rings of  $X$  are factorial, then  $Q^\tau \rightarrow N$  is an isomorphism in the following cases: when  $k(0)$  is perfect and when a certain invariant  $\delta$  is coprime to the residual characteristic.

The proof uses the characterization of the Néron model in terms of the weak Néron mapping property. Recall that a  $S$ -scheme  $Y \rightarrow S$  is said to be

a weak Néron model of its generic fiber if the natural map  $Y(S) \rightarrow Y(\eta)$  is bijective. If  $G \rightarrow S$  is a finite type  $S$ -group scheme whose generic fiber is an abelian variety, then  $G$  is the Néron model of its generic fiber if and only if it satisfies the weak Néron mapping property ([BLR90, Sec. 7.1, Thm. 1]).

### 3. THE MAIN THEOREM

Here we derive the main results for families over a strict henselian discrete valuation ring  $S$ . Specifically, we provide sufficient condition for the maximal separated quotient  $Q^\tau$  of the Picard functor to be the Néron model, and we relate  $Q^\tau$  to a fine moduli space of line bundles that satisfies certain axioms. These moduli spaces are, by definition, subfunctors of a (large) functor that we now define.

**Definition 3.1.** If  $T$  is a  $S$ -scheme, then we define  $\text{Sheaf}(X_T)$  to be the set of isomorphism classes of  $\mathcal{O}_T$ -flat, finitely presented  $\mathcal{O}_{X_T}$ -modules  $\mathcal{I}$  on  $X_T$  that are fiber-wise pure, rank 1, and of degree 0.

The functor  $\text{Sh} = \text{Sh}_{X/S}: S\text{-Sch} \rightarrow \text{Sets}$  is defined to be the étale sheaf associated to the functor

$$(3.1) \quad T \mapsto \text{Sheaf}(X_T).$$

There is a tautological transformation  $P^0 \rightarrow \text{Sh}$  that realizes  $P^0$  as a subfunctor of  $\text{Sh}$ . In fact:

**Lemma 3.2.** *The subfunctor  $P^0 \subset \text{Sh}$  is open.*

*Proof.* Given a  $S$ -scheme  $T$  and a morphism  $g: T \rightarrow \text{Sh}$ , we must show that  $T \times_{\text{Sh}} P^0$  is representable by a scheme and that  $T \times_{\text{Sh}} P^0 \rightarrow T$  is an open immersion. Thus, let  $g$  be given.

By definition, there exists an étale surjection  $p: T' \rightarrow T$  and a sheaf  $\mathcal{I}' \in \text{Sheaf}(X_{T'})$  that represents  $g \circ p: T' \rightarrow \text{Sh}$ . Consider the subset  $U' \subset T'$  of points  $t \in T'$  with the property that the restriction of  $\mathcal{I}'$  to the fiber  $X_t$  is a line bundle. This locus is open by [AK80, Lemma 5.12(a)], and one may easily show that  $U'$  represents  $T' \times_{\text{Sh}} P^0$ . A descent argument establishes the analogous property for the image  $U$  of  $U'$  under  $T' \rightarrow T$ . This completes the proof.  $\square$

A remark about topologies: we work with the étale sheaf associated to Equation 3.1, but one could instead work with the associated fppf sheaf. When  $f$  is projective; it is a theorem of Altman–Kleiman that the subfunctor of  $\text{Sh}$  parameterizing simple sheaves can be represented by a quasi-separated, locally finitely presented  $S$ -algebraic space, and hence is a fppf sheaf ([AK80, Thm. 7.4]). We do not know if  $\text{Sh}$  is an fppf sheaf in general. Here  $\text{Sh}$  is just used as a tool for keeping track of schemes, and certainly any representable subfunctor of  $\text{Sh}$  is a fppf sheaf.

One reason for working with the étale topology instead of the fppf topology is that it makes the following fact easy to prove.

**Fact 3.3.** *The natural map  $\text{Sheaf}(X) \rightarrow \text{Sh}(S)$  is surjective.*



*Proof.* Let  $g \in \text{Sh}(S)$  be given. By definition, there is an étale morphism  $S' \rightarrow S$  and an element  $\mathcal{I}' \in \text{Sheaf}(X_{S'})$  that maps to  $g_{S'} \in \text{Sh}(S')$ . But  $S$  is strict henselian, so  $S' \rightarrow S$  may be taken to be an isomorphism  $S \rightarrow S$  ([EGAIV4, Prop. 18.8.1(c)]), in which case the result is obvious.  $\square$

The following two facts about separably closed fields are standard, but they will be used so frequently that it is convenient to record them.

**Fact 3.4.** *If  $k(0)$  is a separably closed field and  $f_0: Y_0 \rightarrow \text{Spec}(k(0))$  is smooth of relative dimension  $n$ , then the closed points of  $Y_0$  with residue field  $k(0)$  are dense.*

*Proof.* This is [BLR90, Cor. 13]. The scheme  $Y_0$  can be covered by affine opens  $U_0$  that admit an étale morphism  $p: U_0 \rightarrow \mathbb{A}_{k(0)}^n$ . Certainly, the closed points with residue field  $k(0)$  are dense in the image of  $p$ . If  $v_0 \in \mathbb{A}_{k(0)}^n$  is one such point, then  $p^{-1}(v_0)$  is a finite, étale  $k(0)$ -algebra, hence a disjoint union of closed points defined over  $k(0)$ . Density follows.  $\square$

Fact 3.4 is typically used together with the following fact to assert that a smooth morphism has many sections.

**Fact 3.5.** *Let  $Y \rightarrow S$  be a smooth morphism over strict henselian discrete valuation ring. Then  $Y(S) \rightarrow Y(k(0))$  is surjective.*

*Proof.* This is [EGAIV4, Cor. 17.17.3] (or [BLR90, Prop. 14]). If  $U$  and  $X'$  are as in the statement of [EGAIV4, Cor. 17.17.3], then we must have  $U = S$  and  $X' \rightarrow U$  may be taken to be an isomorphism (again, by [EGAIV4, Prop. 18.8.1(c)]).  $\square$

We now prove the main results of the paper.

**Proposition 3.6.** *Let  $f: X \rightarrow S$  be a family of curves and  $J \subset \mathbb{P}^0$  a subfunctor such that the generic fibers  $J_\eta = \mathbb{P}_\eta^0$  coincide. Assume  $J$  is represented by a smooth, finitely presented  $S$ -scheme.*

*If  $J$  is  $S$ -separated, then  $J \rightarrow \mathbb{Q}$  is an open immersion. Furthermore, the image is contained in  $\mathbb{Q}^\tau$  provided  $\mathbb{Q}^\tau \subset \mathbb{Q}$  is closed (e.g., the local rings of  $X$  are factorial).*

*Proof.* This is an application of Zariski's Main Theorem. We begin by showing that the induced map  $J \rightarrow \mathbb{Q}$  is injective on closed points. It is enough to verify this after extending base  $S$  so that  $k(0)$  is algebraically closed. Thus, we will temporarily assume  $k := k(0)$  is algebraically closed and work with  $k$ -valued points instead of closed points. Given  $q \in \mathbb{Q}(k)$ , there is nothing to show when the fiber over  $q$  is empty. If non-empty, pick  $p \in J(k)$  mapping to  $q$ . We may invoke Fact 3.5 to assert that there exists a section  $\sigma \in J(S)$  with  $\sigma(0) = p$ .

The fiber of  $q$  under  $\mathbb{P} \rightarrow \mathbb{Q}$  is the set of elements of the form  $p + e$  with  $e \in E(k)$  or, equivalently, the elements of  $(\sigma + E)(k)$  ([Ray70, Cor. 4.1.2]). Restricting to  $J$ , we see that the fiber of  $q$  under  $J \rightarrow \mathbb{Q}$  is the set of  $k$ -valued

points of  $(\sigma + E) \cap J$ . But  $(\sigma + E) \cap J$  is the scheme-theoretic closure of  $\sigma$  in  $J$  (by [EGAIV2, 2.8.5]), which is just the image of  $\sigma$  by separatedness. In particular, the pre-image of  $q$  under  $J \rightarrow Q$  must be the singleton set  $\{p\}$ . This proves that the map is injective on closed points. We now return to the case where  $S$  is a henselian discrete valuation ring (so  $k(0)$  is no longer assumed to be algebraically closed).

It follows that the set-theoretic fibers of  $J \rightarrow Q$  are finite sets. Indeed, if  $Z \subset J$  is the locus of points  $x \in J$  with the property that  $x$  lies in a positive dimensional fiber, then  $Z$  is closed by Chevalley's Theorem ([EGAIV2, 13.1.3]). Furthermore,  $Z$  is contained in the special fiber  $J_0$  and contains no closed points. This is only possible if  $Z$  is the empty scheme. In other words, the set-theoretic fibers of  $J \rightarrow Q$  are 0-dimensional, hence finite (by [EGAIV1, 14.1.9]).

It follows immediately from Zariski's Main Theorem ([EGAI, 4.4.9]) that  $J \rightarrow Q$  is an open immersion. This proves the first part of the theorem. To complete the proof, observe that flatness implies that the generic fiber of  $J_\eta$  is dense in  $J$  ([EGAIV2, 2.8.5]). In particular,  $J$  is contained in the closure of  $J_\eta$  in  $Q$ . The generic fiber of  $J$  coincides with the generic fiber of  $Q^\tau$ , so the closure of this common scheme is contained in  $Q^\tau$  when  $Q^\tau \subset Q$  is closed. This completes the proof.  $\square$

**Remark 3.7.** In Proposition 3.6, we do *not* assume that  $J \subset P^0$  is an open subfunctor, but this condition holds in most cases of interest. When open,  $J$  is automatically formally smooth and locally of finite presentation. Thus, the key hypothesis in the proposition is that  $J$  is represented by a  $S$ -separated scheme. A similar remark holds for Theorem 3.10; there the key hypotheses are that  $\bar{J}$  satisfies the valuative criteria of properness and that  $J$  is representable. Indeed, we do not even need to assume that  $\bar{J}$  is representable.

Under stronger assumptions, we can actually show that the natural map  $J \rightarrow Q^\tau$  is an isomorphism. The essential point is to prove that  $J$  satisfies the weak Néron mapping property. When  $J$  can be embedded in a  $S$ -proper moduli space  $\bar{J}$ , this property holds provided that the local rings of  $X$  are factorial. The content of this claim is that a line bundle on the generic fiber can specialize only to a line bundle on the special fiber. By localizing, the claim is equivalent to the following lemma, which is based on a proof from [AK79] (p. 27 after “Step XII”).

**Lemma 3.8.** *Suppose that  $(R, \pi)$  is a discrete valuation ring and  $R \rightarrow \mathcal{O}$  a local, flat algebra extension with  $\mathcal{O}$  noetherian. Let  $M$  be a  $R$ -flat, finite  $\mathcal{O}$ -module with the property that  $M[\pi^{-1}]$  is free of rank 1 and  $M/\pi M$  is a rank 1, pure module. If  $\mathcal{O}$  is factorial, then  $M$  is free of rank 1.*

*Proof.* We can certainly assume  $\mathcal{O}$  is not the zero ring. To ease notation, we write  $\bar{M} := M/\pi M$  and  $\bar{\mathcal{O}} := \mathcal{O}/\pi\mathcal{O}$ . It is enough to prove that  $M$

is isomorphic to a height 1 ideal. Indeed, such an ideal is principal by the factoriality assumption.

We argue by first showing that  $M$  is isomorphic to an ideal of  $\mathcal{O}$ . Let  $\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_n$  be the minimal primes of  $\bar{\mathcal{O}}$  and  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  the corresponding primes of  $\mathcal{O}$ . We have assumed that the stalk  $\bar{M} \otimes k(\bar{\mathfrak{p}}_i)$  is 1-dimensional. This stalk coincides with the stalk  $M \otimes k(\mathfrak{p}_i)$ , so we can conclude that the localization  $M_{\mathfrak{p}_i}$  is free of rank 1 for  $i = 1, \dots, n$ .

We can also conclude that the same holds for the localizations of the dual module  $M^\vee := \text{Hom}(M, \mathcal{O})$ . An application of the Prime Avoidance Lemma shows that there exists an element  $\phi \in M^\vee$  that maps to a generator of  $M_{\mathfrak{p}_i}^\vee$  for all  $i$ . We will show that  $\phi: M \rightarrow \mathcal{O}$  realizes  $M$  as a  $R$ -flat family of ideals (i.e.,  $\phi$  is injective with  $R$ -flat cokernel).

It is enough to show that the reduction  $\bar{\phi}: \bar{M} \rightarrow \bar{\mathcal{O}}$  is injective. This map factors as

$$\bar{M} \rightarrow \oplus \bar{M}_{\bar{\mathfrak{p}}_i} \rightarrow \oplus \bar{\mathcal{O}}_{\bar{\mathfrak{p}}_i}.$$

The kernel of the left-most map is a submodule whose support does not contain any generic point of a component of  $\text{Spec}(\mathcal{O})$ , and thus must be zero by pureness. Furthermore, the right-most map is an isomorphism by construction.

Now consider the ideal  $I[\pi^{-1}]$  given by the image of  $\phi[\pi^{-1}]: M[\pi^{-1}] \rightarrow \mathcal{O}[\pi^{-1}]$ . This is a principal ideal, hence is either the unit ideal or an ideal of height at most 1 (Hauptidealsatz!). By flatness, the same is true of the image  $I$  of  $\phi$ . In fact,  $I$  cannot be a height zero ideal: the only such prime is the zero ideal, which does not satisfy the hypotheses. Thus  $I$  is either the unit ideal or a height 1 ideal. In either case,  $I$  must be principal, and the proof is complete.  $\square$

**Remark 3.9.** In Lemma 3.8, the hypothesis that  $\bar{M}$  is rank 1 cannot be removed. Indeed, the rank 1 condition is automatic when  $\bar{\mathcal{O}}$  is reduced, but not in general. Consider  $\mathcal{O} := k[[x, \epsilon, \pi]]/(\epsilon^2 = \pi)$ ,  $R := k[[\pi]]$ , and  $M := \epsilon \cdot \mathcal{O} + t \cdot \mathcal{O}$ . All the hypotheses of the lemma are satisfied except that  $\bar{M}$  is not rank 1 (it is isomorphic to  $\bar{\mathcal{O}}_{\text{red}}^{\oplus 2}$ ). Consequently,  $M$  is not free of rank 1. This example is taken from [EG95, p. 765].

The factorial condition is important in what follows, so we record it as a hypothesis.

**Hypothesis A.** *We say a family of curves  $f: X \rightarrow B$  over a Dedekind scheme satisfies Hypothesis A if the generic fiber  $X_\eta$  is smooth and the local rings of  $X_S$  are factorial for every strict henselization  $S \rightarrow B$ .*

Observe Hypothesis A is satisfied when  $X$  is regular and  $X_\eta$  is smooth. We now prove the main theorem of this paper.

**Theorem 3.10.** *Let  $f: X \rightarrow S$  be a family of curves and  $\bar{J}$  a subfunctor of  $\text{Sh}$  such that the generic fibers  $\bar{J}_\eta = \text{Sh}_\eta$  coincide. Assume the line bundle locus  $J \subset \bar{J}$  is represented by a smooth and finitely presented  $S$ -scheme.*

If  $\bar{J}$  satisfies the valuative criteria of properness and  $f$  satisfies Hypothesis A, then  $Q^\tau$  is the Néron model and

$$J \subset Q^\tau = N$$

is an open subscheme that contains all the  $k(0)$ -valued points of  $Q^\tau$ . Furthermore,

$$J = Q^\tau = N$$

provided one of the following conditions hold:

- (1)  $k(0)$  is algebraically closed;
- (2)  $J$  is stabilized by the identity component  $Q^o$ .

*Proof.* By Proposition 3.6, the natural map  $J \rightarrow Q$  is an open immersion with image contained in  $Q^\tau$ . Using this fact, we can prove that  $Q^\tau$  is the Néron model of its generic fiber. Indeed, it is enough to prove that  $Q^\tau$  satisfies the weak Néron mapping property. The open subscheme  $J \subset Q^\tau$ , in fact, already satisfies this property. Let  $\sigma_\eta \in Q^\tau(\eta) = J(\eta)$  be given. By properness, we can extend  $\sigma_\eta$  to a section  $\sigma \in \bar{J}(S)$ , and this element can be represented by a family  $\mathcal{I}$  of pure, rank 1 sheaves (by Fact 3.3). But every such family is a family of line bundles (Lemma 3.8), and hence  $\sigma$  lies in  $J(S) \subset \bar{J}(S)$ . In other words,  $J$  satisfies the weak Néron mapping property.

The weak Néron mapping property of  $J$  also implies that the image of  $J$  contains all the  $k(0)$ -valued points of  $Q^\tau$ . Indeed, every  $k(0)$ -valued point of  $Q^\tau$  is the specialization of a section by Fact 3.5. If  $k(0)$  is algebraically closed, then we have shown that  $J$  contains every  $k(0)$ -valued point of  $Q^\tau$ , hence every closed point. Thus,  $J = Q^\tau$ , and there is nothing more to show.

Let us now turn our attention to the case where  $k(0)$  is only separably closed, but  $J$  is stabilized by  $Q^o$ . Our goal is to show  $J = Q^\tau$ , and to show this, we pass to the special fiber  $J_0 \rightarrow Q_0^\tau$  and argue with points. Let  $x$  be a  $\bar{k}(0)$ -valued point of  $Q^\tau$ , where  $\bar{k}(0)$  is the algebraic closure of the residue field. By density (Fact 3.4), there exists a  $\bar{k}(0)$ -valued point  $y$  in the image of  $J_0 \rightarrow Q_0^\tau$  that lies in the same connected component as  $x$ . We have  $x = y + (x - y)$ , which expresses  $x$  as the sum of a point of  $Q_0^o$  and a point of  $J_0$ . The point  $x$  must lie in  $J_0$  by assumption. This shows that the image of  $J$  contains all of  $Q^\tau$ , completing the proof.  $\square$

**Remark 3.11.** The hypothesis that  $J$  is stabilized by the identity component  $Q^o$  is perhaps unexpected, but it is often satisfied in practice. The moduli space  $\bar{J}$  is typically constructed by imposing numerical conditions on the multi-degree of a sheaf, and the multi-degree is invariant under the action of  $Q^o$  (because the action is given by tensoring with a multi-degree 0 line bundle).

In the next section, we will show that certain moduli spaces constructed in the literature satisfy the hypotheses of Theorem 3.10. There are, however, families of curves  $f: X \rightarrow S$  with factorial local rings  $\mathcal{O}_{X,x}$  such that there does not exist a  $S$ -scheme  $\bar{J}$  satisfying the conditions of the theorem. Indeed,

the family  $f: X \rightarrow S$  in [Ray70, Ex. 9.2.3] is a family of genus 1 curves such that the local rings of  $X$  are factorial (even regular), but the natural map  $Q^7 \rightarrow N$  is not an isomorphism. In particular, no  $\bar{J}$  satisfying the hypotheses of Theorem 3.10 can exist.

#### 4. APPLICATIONS

Here we apply Theorem 3.10 to some families of moduli spaces from the literature and then deduce consequences. The two moduli spaces that we are interested in are the Esteves moduli space of quasi-stable sheaves (Sect. 4.1) and the Simpson moduli space of slope stable sheaves (Sect. 4.2). In Section 4.3, we discuss the special case of families of genus 1 curves, where suitable moduli spaces can be constructed explicitly.

The moduli spaces we study are associated to a relatively projective family of curves. We are primarily interested in families over a Dedekind scheme with locally factorial total space, in which case projectivity is automatic. This fact is a consequence of the generalized Chevalley Conjecture when the Dedekind scheme is defined over a field, but we do not know a reference in general. For completeness, we prove:

**Proposition 4.1.** *Let  $f: X \rightarrow B$  be a family of curves over a Dedekind scheme. If the local rings of  $X$  are factorial, then  $f$  is projective.*

*Proof.* This proof was explained to the author by Steven Kleiman. Fix a closed point  $b_0 \in B$ . Given any component  $F \subset X_{b_0}$ , I claim that we can find a line bundle  $\mathcal{L}$  on  $X$  that has non-negative degree on every component of every fiber and strictly positive degree on  $F$ .

Pick a closed point  $x \in F$  and an open affine neighborhood  $U \subset X$  of that point. By the Prime Avoidance Lemma, we can find a regular function  $r \in H^0(U, \mathcal{O}_X)$  that does not vanish on any component of  $X_{b_0}$  but does vanish at  $x$ . Pick a component  $D$  of the closure of  $\{r = 0\} \subset U$  in  $X$ . Then  $D$  is a Cartier divisor (by the factoriality assumption) that does not contain any component of any fiber  $X_b$  (by construction). Furthermore,  $D$  has non-trivial intersection with  $F$ . The associated line bundle  $\mathcal{L} := \mathcal{O}_X(D)$  has the desired positivity property.

Now construct one such line bundle for every irreducible component  $F$  of  $X_{b_0}$  and define  $\mathcal{M}$  to be their tensor product. The line bundle  $\mathcal{M}$  is nef on every fiber and ample on  $X_{b_0}$ . Ampleness is an open condition, so  $\mathcal{M}$  is in fact ample on all but finitely many fibers of  $f$ . After repeating the construction for each such fiber and forming the tensor product, we have constructed a  $f$ -relatively ample line bundle on  $X$ . This completes the proof.  $\square$

We now turn our attention to the moduli spaces.

**4.1. Esteves Jacobians.** We first discuss the Esteves moduli space of quasi-stable sheaves. This moduli space fits very naturally into the framework of the previous section.

Suppose  $B$  be a locally noetherian scheme and  $f: X \rightarrow B$  a projective family of curves whose fibers are geometrically reduced. Quasi-stability is defined in terms of a section  $\sigma: B \rightarrow X^{\text{sm}}$  and a vector bundle  $\mathcal{E}$  on  $X$  with fiber-wise integral slope  $\deg(\mathcal{E}_b)/\text{rank}(\mathcal{E}_b)$ , which we assume is constant as a function of  $b \in B$ . Given  $\sigma$  and  $\mathcal{E}$ ,  $\sigma$ -quasi-stability is a numerical condition on the multi-degree of a rank 1, torsion-free sheaf of degree

$$d(\mathcal{E}) = d := -\chi(\mathcal{O}_{X_b}) - \deg(\mathcal{E}_b)/\text{rank}(\mathcal{E}_b).$$

For the definitions (which we will not use), we direct the reader to [Est01, p. 3051] (for a single sheaf) and [Est01, p. 3054] (for a family). The basic existence theorem is Theorem A (stated on [Est01, p. 3047]; proved in [Est01, Sect. 4]). It states that if  $\text{Sheaf}_{\mathcal{E}}^{\sigma}: B\text{-Sch} \rightarrow \text{Sets}$  is the functor defined by setting  $\text{Sheaf}_{\mathcal{E}}^{\sigma}(T)$  equal to the set of isomorphism classes of  $\mathcal{O}_T$ -flat, finitely presented  $\mathcal{O}_{X_T}$ -modules that are fiber-wise  $\sigma$ -quasi-stable, then there is a  $B$ -proper algebraic space  $\bar{J}_{\mathcal{E}}^{\sigma} \rightarrow B$  of finite type that represents the étale sheaf associated to  $\text{Sheaf}_{\mathcal{E}}^{\sigma}$ .

Strictly speaking, our definition differs from the one given in [Est01] in two ways. First, in [Est01] the author does not work with isomorphism classes of sheaves but rather with equivalence classes under the relation given by identifying two sheaves  $\mathcal{I}_1$  and  $\mathcal{I}_2$  on  $X_T$  when  $\mathcal{I}_1$  is isomorphic to  $\mathcal{I}_2 \otimes f^*(\mathcal{L})$  for some line bundle  $\mathcal{L}$  on  $T$ . Zariski locally on  $T$  the sheaves  $\mathcal{I}_1$  and  $\mathcal{I}_2$  are isomorphic, and it follows that the étale sheaf associated to  $\text{Sheaf}_{\mathcal{E}}^{\sigma}$  is canonically isomorphic to the sheaf considered in [Est01]. Second, Esteves only defines his functor as a functor from locally noetherian schemes to sets. However, the functor  $\text{Sheaf}_{\mathcal{E}}^{\sigma}$  and its associated étale sheaf are easily seen to be locally finitely presented. It follows that  $\bar{J}_{\mathcal{E}}^{\sigma}$  represents the étale sheaf associated to  $\text{Sheaf}_{\mathcal{E}}^{\sigma}$ , rather than just the restriction of this sheaf to locally noetherian schemes.

If  $f$  satisfies stronger conditions, then the space  $\bar{J}_{\mathcal{E}}^{\sigma}$  is actually a scheme. This is the content of Theorem B (stated on [Est01, p. 3048]; proved on [Est01, p. 3086]). The theorem states that if there exist sections  $\sigma_1, \dots, \sigma_n: B \rightarrow X^{\text{sm}}$  of  $f$  with the property that every irreducible component of a fiber  $X_b$  is geometrically integral and contains one of the points  $\sigma_1(b), \dots, \sigma_n(b)$ , then  $\bar{J}_{\mathcal{E}}^{\sigma}$  is a scheme.

In the special case where  $B = S$  is a strict henselian discrete valuation ring, the hypotheses of Theorem B are automatically satisfied. Indeed, the locus of  $k(0)$ -valued points is dense in the smooth locus  $X_0^{\text{sm}}$  (Fact 3.4), which in turn is dense in  $X_0$  as  $X_0$  is geometrically reduced. We may conclude that the irreducible components of  $X_0$  are geometrically integral. Finally, every  $k(0)$ -valued point of  $X_0$  extends to a section  $\sigma: S \rightarrow X$  (Fact 3.5), so the hypotheses to Theorem B are certainly satisfied.

We call  $\bar{J}_{\mathcal{E}}^{\sigma}$  the **Esteves compactified Jacobian**. Inside of the Esteves compactified Jacobian, we can consider the open subscheme parameterizing line bundles. This scheme is called the **Esteves Jacobian** and denoted by  $J_{\mathcal{E}}^{\sigma}$ . While the scheme  $\bar{J}_{\mathcal{E}}^{\sigma}$  parameterizes sheaves, it is not naturally a

subfunctor of  $\text{Sh}$  because it does not parameterize degree 0 sheaves. We can, however, define a natural transformation  $\bar{J}_{\mathcal{E}}^{\sigma} \rightarrow \text{Sh}$  by the rule

$$\mathcal{I} \mapsto \mathcal{I}(-d \cdot \sigma)$$

Both Proposition 3.6 and Theorem 3.10 apply to  $\bar{J}_{\mathcal{E}}^{\sigma}$ .

**Corollary 4.2.** *Fix a Dedekind scheme  $B$ . Let  $f: X \rightarrow B$  be a projective family of geometrically reduced curves. Let  $\sigma: B \rightarrow X^{\text{sm}}$  be a section and  $\mathcal{E}$  a vector bundle on  $X$  with fiber-wise integral slope.*

*Then the natural map  $J_{\mathcal{E}}^{\sigma} \rightarrow \mathcal{Q}$  is an open immersion. Assume further:*

- *$f$  satisfies Hypothesis A.*

*Then  $J_{\mathcal{E}}^{\sigma} = \mathcal{Q}^{\tau}$ , and this scheme is the Néron model.*

*Proof.* By localizing, we can assume that  $B = S$  is a strict henselian discrete valuation ring, in which case we are reduced to proving that the hypotheses of Proposition 3.6 and Theorem 3.10 hold. The scheme  $J_{\mathcal{E}}^{\sigma}$  is easily seen to be formally  $S$ -smooth. Indeed,  $\sigma$ -quasi-stability is a condition on fibers, so the formal smoothness of  $P^0$  implies the formal smoothness of  $J_{\mathcal{E}}^{\sigma}$ . The remaining hypotheses of Proposition 3.6 are explicitly assumed, so we can deduce the first part of the theorem. To complete the proof, it is enough to show that  $J_{\mathcal{E}}^{\sigma}$  is stabilized by  $\mathcal{Q}^0$ . But the action of  $\mathcal{Q}^0$  on  $J_{\mathcal{E}}^{\sigma}$  is given by the tensor product against a multi-degree 0 line bundle, so this action preserves multi-degree and hence  $\sigma$ -quasi-stability.  $\square$

Corollary 4.2 implies that  $J_{\mathcal{E}}^{\sigma}$  is a scheme with (unique)  $B$ -group scheme structure that extends the group scheme structure of the generic fiber. It is not immediate from the definition that  $J_{\mathcal{E}}^{\sigma}$  admits such structure, and Example 4.9 shows that the group structure is special to the case of families over a 1-dimensional base. The result also implies uniqueness results for the Esteves Jacobian; if  $\sigma': B \rightarrow X^{\text{sm}}$  is a second section and  $\mathcal{E}'$  a second vector bundle on  $X$ , then  $J_{\mathcal{E}'}^{\sigma'}$  is canonically isomorphic to  $J_{\mathcal{E}}^{\sigma}$ . In the next section, we will define the Simpson stable Jacobian  $J_{\mathcal{L}}^0(X)$ , and this scheme is also isomorphic to  $J_{\mathcal{E}}^{\sigma}$  provided every slope semi-stable sheaf is stable. It would be interesting to know if these isomorphisms extend to the compactifications. Important results along these lines can be found in [MV10] and [Est09], but many basic question remain unanswered. Currently, there is not a single example of a curve  $X_0 \rightarrow \text{Spec}(k)$  such that two Esteves compactified Jacobians associated to  $X_0$  are non-isomorphic.

**4.2. Simpson Jacobians.** The hypotheses to Proposition 3.6 and Theorem 3.10 are satisfied by certain moduli spaces of stable sheaves, which we call Simpson Jacobians. Here we recall Simpson's construction, along with later work of Langer and Maruyama, and then apply results from Section 3. We restrict our attention to families of reduced curves (but see Remark 3.9, Remark 4.4, and the discussion preceding Example 4.9).

We work over a scheme  $B$  that is finitely generated over a universally Japanese ring  $R$  (e.g.  $R = \mathbb{C}, \mathbb{F}_p, \mathbb{Z}, \dots$ ). Let  $f: X \rightarrow B$  a family of curves

with  $f$ -relatively ample line bundle  $\mathcal{L}$ , and assume the Euler–Poincaré characteristics  $\chi(\mathcal{O}_{X_b})$  and  $\chi(\mathcal{L}_b)$  are constant as functions of the base  $B$ . Set  $P_d$  equal to the polynomial

$$(4.1) \quad P_d(t) := \deg(\mathcal{L}_b) \cdot t + d + \chi,$$

where  $\chi$  is the Euler–Poincaré characteristic of a fiber of  $f$  and  $\deg(\mathcal{L}_b)$  is the degree of the restriction of  $\mathcal{L}$  to a fiber. This is the Hilbert polynomial of a degree  $d$  line bundle.

Given this data, Simpson constructed an associated moduli space provided  $R = \mathbb{C}$ . The Simpson moduli space  $M(\mathcal{O}_X, P_d)$  parameterizes slope semi-stable sheaves with Hilbert polynomial  $P_d$ . (See [Sim94, p. 54–56] for the definition of semi-stability). To be precise, define  $M^\sharp(\mathcal{O}_X, P_d)$  to be the functor whose  $T$ -valued points are isomorphism classes of  $\mathcal{O}_T$ -flat, finitely presented  $\mathcal{O}_{X_T}$ -modules whose fibers are  $\mathcal{L}$ -slope semi-stable sheaves with Hilbert polynomial  $P_d$ . The main existence result ([Sim94, Thm. 1.21]) asserts that there is a projective scheme  $M(\mathcal{O}_X, P_d)$  that corepresents  $M^\sharp(\mathcal{O}_X, P_d)$ . Inside of  $M(\mathcal{O}_X, P_d)$ , we may consider the open subscheme  $M^{\text{st}}(\mathcal{O}_X, P_d)$  parameterizing  $\mathcal{L}$ -slope stable sheaves. The stable locus is a fine moduli space: its  $\mathbb{C}$ -valued points are in natural bijection with the isomorphism classes of  $\mathcal{L}$ -slope stable sheaves with Hilbert polynomial  $P_d$ , and étale locally on  $M^{\text{st}}(\mathcal{O}_X, P_d)$ , the product  $X \times_B M^{\text{st}}(\mathcal{O}_X, P_d)$  admits a universal family of sheaves. The reader may check that these conditions are equivalent to the condition that  $M^{\text{st}}(\mathcal{O}_X, P_d)$  represent the étale sheaf associated the functor parameterizing stable sheaves. While Simpson only considers the case  $R = \mathbb{C}$ , later work of Langer ([Lan04a, Thm. 4.1], [Lan04b, Thm. 0.2]) and Maruyama ([Mar96]) extends these results to the case where  $R$  is an arbitrary universally Japanese ring.

Let us now specialize to the case where  $B$  is a Dedekind scheme. When  $f$  has reducible fibers,  $M^{\text{st}}(\mathcal{O}_X, P_d)$  may contain points corresponding to sheaves that are not rank 1 (see [LM05, Ex. 2.2]). This is potentially a major source of confusion: the term “rank” is used in a different way in [Sim94], and the sheaves parameterized by  $M^{\text{st}}(\mathcal{O}_X, P_d)$  are rank 1 in Simpson’s sense but not necessary in the sense used here.

We avoid these sheaves. Define the **Simpson stable Jacobian**  $J_{\mathcal{L}}^d$  of degree  $d$  to be the locus of stable line bundles in  $M^{\text{st}}(\mathcal{O}_X, P_d)$  (which is an open subscheme by [AK80, Lemma 5.12(a)]). We define the **Simpson stable compactified Jacobian**  $\bar{J}_{\mathcal{L}}^d$  to be the subset of the stable locus  $M^{\text{st}}(\mathcal{O}_X, P_d)$  that corresponds to pure, rank 1 sheaves. (Warning: the compactified Jacobian is a  $B$ -proper scheme when every semi-stable pure sheaf with Hilbert polynomial  $P_d$  is stable but not in general!)

When the fibers of  $X \rightarrow B$  are *geometrically reduced*, a minor modification of the proof of [Pan96, Lemma 8.1.1] shows that the subset  $\bar{J}_{\mathcal{L}}^d \subset M^{\text{st}}(\mathcal{O}_X, P_d)$  is closed and open, and hence has a natural scheme structure. We record this.



**Lemma 4.3.** *Assume the fibers of  $f: X \rightarrow B$  are geometrically reduced. Then the subset  $\bar{J}_{\mathcal{L}}^d$  is closed and open in  $M^{\text{st}}(\mathcal{O}_X, P_d)$ .*

*Proof.* The main point to prove is that a 1-parameter family of line bundles cannot specialize to a pure sheaf that fails to have rank 1, and this is shown by examining the leading term of the Hilbert polynomial. To begin, we may cover  $M^{\text{st}}(\mathcal{O}_X, P_d)$  by étale morphisms  $M \rightarrow M^{\text{st}}(\mathcal{O}_X, P_d)$  with the property that a universal family  $\mathcal{I}_{\text{uni.}}$  on  $M \times_B X$  exists. It is enough to verify the claim after passing from  $M^{\text{st}}(\mathcal{O}_X, P_d)$  to an arbitrary such scheme, and so for the remainder of the proof we work with  $M$  in place of  $M^{\text{st}}(\mathcal{O}_X, P_d)$ . We will also abuse notation by denoting the pullback of  $\bar{J}_{\mathcal{L}}^d$  under  $M \rightarrow M^{\text{st}}(\mathcal{O}_X, P_d)$  by the same symbol  $\bar{J}_{\mathcal{L}}^d$ .

We first need to check that  $\bar{J}_{\mathcal{L}}^d \subset M$  is constructible, so that we can make use of the valuative criteria. Given  $m \in M$  mapping to  $b \in B$ , the condition that the fiber  $I_m$  is rank 1 is just the condition that the restriction of  $I_m$  to  $X_b^{\text{sm}}$  is a line bundle. Constructibility thus follows from [EGAIV3, 9.4.7] applied to  $M \times_B X^{\text{sm}} \rightarrow M$ .

To complete the proof, it is enough to prove that  $\bar{J}_{\mathcal{L}}^d$  is closed under specialization and generalization. Thus, we pass from  $M$  to a discrete valuation ring  $T$  mapping to  $M$ . If  $\mathcal{I}$  is the sheaf on  $X_T$  given by the pulback of the universal family, then we need to show that the generic fiber of  $I_{\eta}$  is rank 1 if and only if the special fiber  $I_0$  is.

Consider the leading term of the Hilbert polynomial of these sheaves. On one hand, we assumed that the leading term is  $\deg(\mathcal{L}_0)$ . On the other hand, this term can be computed in terms of the generic ranks of the sheaves (using [AK79, 2.5.1]). Suppose that  $x_1, \dots, x_n$  are the generic points of the special fiber and  $y_1, \dots, y_m$  are the generic points of the generic fiber. We write  $\ell_{x_i}(I_0)$  (resp.  $\ell_{y_i}(I_{\eta})$ ) for the length of the stalk of  $I_0$  at  $x_i$  (resp.  $I_{\eta}$  at  $y_i$ ). The leading term of the common Hilbert polynomial can be expressed as:

$$(4.2) \quad \deg(\mathcal{L}_0) = \deg_{x_1}(\mathcal{L}_0) \cdot \ell_{x_1}(I_0) + \dots + \deg_{x_n}(\mathcal{L}_0) \cdot \ell_{x_n}(I_0)$$

$$(4.3) \quad = \deg_{y_1}(\mathcal{L}_{\eta}) \cdot \ell_{y_1}(I_{\eta}) + \dots + \deg_{y_m}(\mathcal{L}_{\eta}) \cdot \ell_{y_m}(I_{\eta})$$

The degree  $\deg(\mathcal{L}_0)$  is also the sum of the partial degrees  $\deg_{x_i}(\mathcal{L}_0)$ , so Eq. (4.2) remains valid if we replace every  $\ell_{x_i}(I_0)$  with 1, and similarly for Eq. (4.3).

Because the fibers are reduced, the lengths  $\ell_{x_i}(I_0)$  and  $\ell_{y_j}(I_{\eta})$  are just the dimension of the stalk of  $I_0$  at  $x_i$  and the stalk of  $I_{\eta}$  at  $y_j$  respectively. In particular, these numbers are upper semi-continuous.

Suppose first that  $I_0$  is rank 1. By semi-continuity, we have  $\ell_{y_i}(I_\eta) \leq 1$ . If some inequality was strict, say  $\ell_{y_1}(I_\eta) = 0$ , then

$$\begin{aligned} \deg(\mathcal{L}_0) &= \deg_{y_1}(\mathcal{L}_\eta) + \deg_{y_2}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta) \\ &> \deg_{y_2}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta) \\ &\geq \deg_{y_1}(\mathcal{L}_\eta) \cdot \ell_{y_1}(I_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta) \cdot \ell_{y_m}(I_\eta) \\ &= \deg(\mathcal{L}_0). \end{aligned}$$

This is absurd! Thus, we must have  $\ell_{y_i}(I_\eta) = 1$  for all  $y_i$  and  $I_\eta$  is rank 1. Similar reasoning shows that if  $I_\eta$  is rank 1, then  $I_0$  is rank 1.  $\square$

**Remark 4.4.** The hypothesis that the fibers of  $f$  are geometrically reduced is necessary. Indeed, the moduli space  $M^{\text{st}}(\mathcal{O}_X, P_d)$  was described in [CK11] in the case that  $X$  is a non-reduced curve whose reduced subscheme  $X_{\text{red}}$  is smooth and whose nilradical  $\mathcal{N}$  is square-zero (i.e.  $X$  is a ribbon). Using that description it is easy to produce examples where  $\bar{J}_{\mathcal{L}}^d \subset M^{\text{st}}(\mathcal{O}_X, P_d)$  is not closed (e.g. take  $d$  equal to 0,  $X$  to have even genus, and  $X_{\text{red}}$  to have genus 1). The points of the complement in the closure correspond to stable vector bundles on  $X_{\text{red}}$ .

We now apply Proposition 3.6 and Theorem 3.10 to the Simpson Jacobians.

**Corollary 4.5.** *Fix a Dedekind scheme that is finitely generated over a universally Japanese ring. Let  $f: X \rightarrow B$  be a family of geometrically reduced curves. Let  $\mathcal{L}$  be  $f$ -relatively ample line bundle.*

*Then the natural map  $J_{\mathcal{L}}^0(X) \rightarrow \mathbb{Q}^\tau$  is an open immersion. Assume further that both of the following conditions hold:*

- *every  $\mathcal{L}$ -slope semi-stable rank 1, torsion free sheaf of degree 0 is  $\mathcal{L}$ -slope stable;*
- *$f$  satisfies Hypothesis A.*

*Then  $J_{\mathcal{L}}^0(X) = \mathbb{Q}^\tau$  and this scheme is the Néron model.*

*Proof.* The local existence of a universal family ([Sim94, Thm. 2.1(4)]) implies that there is a natural transformation  $\bar{J}_{\mathcal{L}}(X) \rightarrow \text{Sh}$  with the property that  $J_{\mathcal{L}}(X)$  is the pre-image of  $P^0 \subset \text{Sh}$ . Furthermore, the slope stability condition is a fiber-wise condition, so a modification of the argument given in Corollary 4.2 completes the proof.  $\square$

**Remark 4.6.** A minor generalization of Corollary 4.5 can be obtained by allowing for moduli spaces of degree  $d$  lines bundles, with  $d \neq 0$ . If we are given a line bundle  $\mathcal{M}$  on  $X$  with fiber-wise degree  $d$ , then there is an associated map  $J_{\mathcal{L}}^d(X) \rightarrow \mathbb{Q}$  that extends the map on the generic fiber given by tensoring with  $\mathcal{M}^{-1}$ . With only notational changes the previous corollary generalizes to a statement about this map.

Corollary 4.5 is, of course, only of interest when there exists an  $\mathcal{L}$  such that  $\mathcal{L}$ -slope stability coincides with  $\mathcal{L}$ -slope semi-stability. Thus, we ask:

when does such a  $\mathcal{L}$  exist? A comprehensive discussion of this question would require a digression on stability conditions, so we limit ourselves to reviewing known results about a single curve  $X_0$  over an algebraically closed field. When  $X_0$  is integral, the stability condition is vacuous, so every ample  $\mathcal{L}_0$  has the desired property. If  $X_0$  is reducible and has only nodes as singularities, then Melo and Viviani have proven the existence of a suitable  $\mathcal{L}_0$  ([MV10, Prop. 6.4]). Stability conditions on reduced, genus 1 curves were analyzed by López-Martín in [LM05]. She exhibits curves  $X_0$  with the property that there is no  $\mathcal{L}_0$  such that every  $\mathcal{L}_0$ -slope semi-stable, pure, rank 1 sheaf degree 0 is stable, but a suitable  $\mathcal{L}_0$  always exists if one considers sheaves of fixed degree  $d \neq 0$ . Finally, stability conditions for a ribbon were analyzed in [CK11]. On a ribbon, the stability condition is independent of  $\mathcal{L}_0$ , and for rational ribbons, slope stability coincides with slope semi-stability precisely when the genus  $g$  is even. It would be desirable to have a general result asserting (non-)existence of a suitable  $\mathcal{L}_0$ .

**4.3. Genus 1 Curves.** The Néron model of the Jacobian of a genus 1 curve can be quite complicated (e.g. see [LLR04]), but these complications do not arise if the family admits a section. Suppose  $B$  is a Dedekind scheme and  $f: X \rightarrow B$  is a family of curves such that the total space  $X$  is regular and the generic fiber  $X_\eta$  is smooth. If  $\sigma: B \rightarrow X^{\text{sm}}$  is a section contained in the smooth locus, then there is a canonical identification of the smooth locus  $X^{\text{sm}}$  with the Néron model  $N$  of the Jacobian of  $X_\eta$ . Here we examine how this fact fits into the preceding framework.

**Definition 4.7.** Let  $f: X \rightarrow B$  be a family of genus 1 curves over a Dedekind scheme and  $\sigma: B \rightarrow X^{\text{sm}}$  a section contained in the smooth locus. We define a sheaf  $\mathcal{I}_{\text{uni.}}$  on  $X \times_B X$  by the formula

$$(4.4) \quad \mathcal{I}_{\text{uni.}} := \mathcal{I}_\Delta(\pi_1^*(\sigma) + \pi_2^*(\sigma)).$$

Here  $\mathcal{I}_\Delta$  is the ideal sheaf of the diagonal and  $\pi_1, \pi_2: X \times_B X \rightarrow X$  are the projection maps.

The sheaf  $\mathcal{I}_{\text{uni.}}$  determines a transformation  $X \rightarrow \text{Sh}$  that realizes  $X$  as a moduli space of sheaves over itself. Proposition 3.6 and Theorem 3.10 apply to this moduli space.

**Corollary 4.8.** *Fix a Dedekind scheme  $B$ . Let  $f: X \rightarrow B$  be a family of genus 1 curves. Let  $\sigma: B \rightarrow X^{\text{sm}}$  be a section.*

*Then the natural map  $X^{\text{sm}} \rightarrow \mathbb{Q}$  is an open immersion. Assume further:*

- *$f$  satisfies Hypothesis A.*

*Then  $X^{\text{sm}} = \mathbb{Q}^\tau$  and this scheme is the Néron model.*

Let us consider the special case where  $B$  is a discrete valuation ring,  $X$  is a minimal regular surface, and the residue field  $k(0)$  is algebraically closed. The possibilities for the special fiber  $X_0$  are given by the Kodaria–Néron classification ([Kod60] and [Nér64]; see [Sil94, p. 353–354] for a recent exposition). The reduced curves appearing in the classification are the Reduction

Types  $I_n$ , II, III, IV. In these cases, one may show that the induced morphism  $X \rightarrow \text{Sh}$  identifies  $X$  with the Esteves compactified Jacobian  $\bar{J}_{\mathcal{O}}^g$ .

In every remaining case (Reduction Type  $I_n^*$ ,  $II^*$ ,  $III^*$ , or  $IV^*$ ) the morphism  $X \rightarrow \text{Sh}$  is not a special case of the fine moduli spaces discussed in the previous two sections. Indeed, the special fiber  $X_0$  is non-reduced, so the Esteves Jacobian of  $X$  is not defined. In Section 4.2, we reviewed Simpson's moduli space  $M^{\text{st}}(\mathcal{O}_X, P_d)$  of stable sheaves, but the image of  $X \rightarrow \text{Sh}$  cannot be described as a closed subscheme of that space. The reason is that slope stable sheaves are simple, but some fibers of  $\mathcal{I}_{\text{uni}}$  are not simple. Specifically, if  $p_0 \in X_0$  lies on the intersection of two components, then the fiber of  $\mathcal{I}_{\text{uni}}$  of  $p_0$  fails to be simple. This can be seen as follows. This fiber is the sheaf  $\mathcal{I}_{p_0}(-\sigma(0))$ , where  $\mathcal{I}_{p_0}$  is the ideal of  $p_0$ . If  $\nu: X'_0 \rightarrow X_0$  is the blow-up of  $X_0$  at  $p_0$ , then one may show that  $H^0(X'_0, \mathcal{O}_{X'_0})$  is canonically isomorphic to the endomorphism ring of  $\mathcal{I}_{p_0}(-\sigma(0))$ . An inspection of the Kodaira–Néron table shows that  $X'_0$  is disconnected, so  $H^0(X'_0, \mathcal{O}_{X'_0})$  does not equal  $k(0)$  and  $\mathcal{I}_{p_0}(-\sigma(0))$  is not simple.

Corollary 4.8 provides a partial answer to a question posed in the introduction: What are the maximal subfunctors  $J$  of  $P^0$  represented by a separated  $B$ -scheme? When  $X$  is, say, regular, a strong result one could hope for is that there is always a subfunctor  $\bar{J}$  of  $\text{Sh}$  satisfying the hypotheses of Theorem 3.10. The line bundle locus  $J \subset \bar{J}$  in such a functor has the property that  $J \rightarrow Q^\tau$  is an isomorphism, and hence  $J$  is maximal. Corollary 4.8 shows that such a  $\bar{J}$  exists when  $f: X \rightarrow S$  is a family of genus 1 curves that  $f$  admits a section. Similarly, the Esteves compactified Jacobian represents a suitable subfunctor when  $f$  has geometrically reduced fibers and admits a section. In general, however, the hope is too optimistic: Raynaud's family, mentioned at the end of Section 3, has that property that no such  $\bar{J}$  can exist.

The question of describing maximal subfunctors  $J$  is most interesting when  $f$  has non-reduced fibers. The slope stable line bundles form a subfunctor  $J \subset P^0$  represented by a  $S$ -separated scheme, but our discussion of genus 1 families together with Remarks 3.9 and 4.4 suggest that we should consider other methods for constructing a suitable  $J$  when  $f$  has non-reduced fibers.

In a different direction, one nice property of the moduli spaces described by Corollary 4.8 is that their geometry is very simple. We use these spaces to provide an example showing that a family  $J \rightarrow B$  of Esteves Jacobians over a regular 2-dimensional base may not have group scheme structure.

**Example 4.9** (Néron models in 2-dimensional families). We will construct a 2-dimensional family  $f: X \rightarrow B$  of plane cubics and an associated Esteves Jacobian  $J \rightarrow B$  with the property that the group law on the locus  $J_U \rightarrow U$  parameterizing non-singular cubics does not extend over all of  $B$ . Furthermore, the family is constructed in such a way that a dense open subset of  $B$  is covered by non-singular curves  $C$  with the property that the restriction

$X_C$  of  $X$  to  $C$  is regular, so  $J_C$  is the Néron model of its generic fiber (and in particular admits group scheme structure that extends the group scheme structure over  $C \cap U$ ). Thus, the Néron models fit into a 2-dimensional family, but their group scheme structure does not.

The idea is as follows. The family we construct has a reducible element  $X_{b_0} \rightarrow b_0$  with the property that, for every non-singular curve  $C \subset B$  passing through  $b_0$  such that  $X_C$  is regular, the restriction of the Esteves Jacobian  $J_C$  is the Néron model of its generic fiber. The fiber  $J_{b_0}$  inherits a group law from this Néron model, and we show explicitly that this group law depends on the particular choice of  $C$ . But, if the group law on  $J_U$  extended to  $J$ , then all the different group laws on  $J_{b_0}$  coming from the different curves  $C$  would be the restriction of one common group law on  $J$ , which is absurd. We now construct the family.

We work over an algebraically closed field  $k$ . The family  $X \rightarrow B$  will be a net of plane cubics. Let  $X_0 \subset \mathbb{P}_k^2$  be a reducible plane cubics that is the union of a smooth quadric  $Q_0$  and a line  $L_0$  that meet in two distinct points. (See Fig. 1.) Fix two general points  $p_1, p_2 \in L_0(k)$  on the line and one general point  $q_1 \in Q_0(k)$  on the quadric. Say that  $F \in H^0(\mathbb{P}_k^2, \mathcal{O}(3))$  is an equation for  $X_0$  and  $G, H \in H^0(\mathbb{P}_k^2, \mathcal{O}(3))$  are two general cubic equations that vanish on all of the points  $p_1, p_2, q_1$ . We will work with the net  $V := \langle F, G, H \rangle \subset H^0(\mathbb{P}_k^2, \mathcal{O}(3))$  and the associated family of curves

$$(4.5) \quad X := \{(p, [r, s, t]) : r \cdot F(p) + s \cdot G(p) + t \cdot H(p) = 0\} \\ \subset \mathbb{P}_k^2 \times \mathbb{P}_k^2.$$

There are two obvious morphisms  $e, f: X \rightarrow \mathbb{P}_k^2$  given by the two projections. If we set  $B := \mathbb{P}_k^2$  equal to the plane, then the second morphism  $f: X \rightarrow B$  realizes  $X$  as a family of genus 1 curves with  $X_0 = f^{-1}(b_0)$ ,  $b_0 := [1, 0, 0]$ . Corresponding to the points  $p_1, p_2, q_1 \in X_0(k)$  are three sections  $\sigma_1, \sigma_2, \tau_1: B \rightarrow X^{\text{sm}}$ , which lie in the smooth locus by the generality assumption.

Another application of the generality assumption shows that the fibers of  $f$  are reduced, so we can form the Esteves Jacobian  $J := J_{\mathcal{E}}^{\sigma_1}$ , where  $\mathcal{E} = \mathcal{O}_X$ . If  $\mathcal{L}_0$  is a line bundle on  $X_0$ , then the quasi-stability condition is that the bidegree  $(\deg(\mathcal{L}_{L_0}), \deg(\mathcal{L}_{Q_0}))$  equals  $(0, 0)$  or  $(1, -1)$ . Now we assume  $J \rightarrow B$  is a group scheme and derive a contradiction.

Suppose that we are given a general line  $C \subset B$  in the plane that contains  $b_0$ . Such a line corresponds to a 2-dimensional linear subspace of the form  $W := \langle F, G_C \rangle \subset V$  for some  $G_C \in V$ . Invoking generality again, the base locus

$$(4.6) \quad \{p \in \mathbb{P}_k^2 : F(p) = G_C(p) = 0\}$$

consists of 9 distinct points. The first projection map  $e: X \rightarrow \mathbb{P}_k^2$  realizes  $X_C$  as the blow-up of the plane at these points, so  $X_C$  is regular, and thus  $J_C$  is the Néron model of its generic fiber. We now study the group of sections of  $J_C \rightarrow C$ .

The base locus (4.6) includes the points  $p_1, p_2, q_1$ . In addition to the points  $p_1, p_2$ , a unique third point of the base locus must lie on the line  $L_0$ . Let us label that point  $p_C$  and write  $\sigma_C: C \rightarrow X^{\text{sm}}$  for the corresponding section.

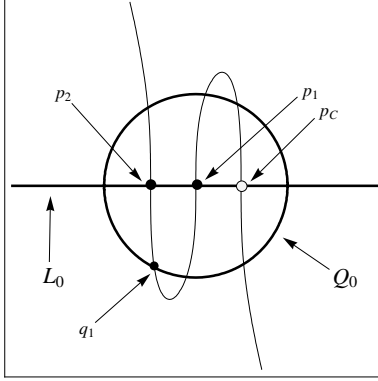


FIGURE 1. The pencil  $X_C$ .

Now consider the following line bundles on  $X_C$ :

$$\begin{aligned}\mathcal{L}_1 &:= \mathcal{O}_X(\sigma_1 - \tau_1), \\ \mathcal{L}_2 &:= \mathcal{O}_X(\sigma_2 - \tau_1), \\ \mathcal{L}_C &:= \mathcal{O}_X(\sigma_C - \tau_1), \\ \mathcal{M} &:= \mathcal{O}_X(1) \otimes \mathcal{O}_X(-3 \cdot \tau_1).\end{aligned}$$

These line bundles are all  $\sigma_1$ -quasi-stable. If we let  $g_1, g_2, g_C, h \in J_C(C)$  respectively correspond to  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_C, \mathcal{M}$ , then I claim we have

$$(4.7) \quad g_1 + g_2 + g_C = h.$$

Indeed, it is enough to verify the claim after passing to the generic fiber of  $J \rightarrow C$ , where the equation is just the statement that the points  $p_1, p_2, p_C$  all lie on a line (the line  $L_0$ ). Now suppose that  $J \rightarrow \mathbb{P}_k^2$  admits a group law extending the group law of the generic fiber. Then the specialization of Equation (4.7) to  $J_{b_0}$  holds for all  $C$  simultaneously. In particular, the isomorphism class of the line bundle  $\mathcal{O}_{X_{b_0}}(p_C - q_1)$  is independent of the particular line  $C \subset \mathbb{P}_k^2$  chosen. But this is absurd: for distinct general lines  $C_1, C_2$ , the points  $p_{C_1}$  and  $p_{C_2}$  (and hence the associated line bundles) are distinct! This completes our discussion of this example.

This example is particularly interesting in light of [OS79]. The authors of that paper consider the case of a family of nodal curves  $f: X \rightarrow B$  over a suitable Dedekind scheme with the property that  $X$  is regular. Let  $J_\eta$  be the Jacobian of the generic fiber. Given a closed point  $0 \in B$ , they prove that the special fiber  $N_0$  of the Néron model of  $J_\eta$  depends only on the curve  $X_0$  and not the particular family  $f$  ([OS79, Cor. 14.4]). This result

must be interpreted with care: in our example, the group law depends on a particular choice of family, but any two such group laws define isomorphic group schemes.

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